Quantum Squeezing Micron-Sized Cantilevers

M. P. Blencowe[1], and M. N. Wybourne[2]
(1) *The Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom*
(2) *Department of Physics and Astronomy, Dartmouth College, Hanover, New Hampshire 03755-3528*
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We show that substantial quantum squeezing of mechanical motion can be achieved for micron-sized cantilever devices fabricated using available techniques. A method is also described for measuring the cantilever fluctuation magnitudes in the squeezing regime.

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Squeezed states—minimum uncertainty states of a harmonic system where the uncertainty of one of the quadrature amplitudes is reduced below that of the zero-point fluctuations (i.e., ground state)—first came to prominence in the late seventies and early eighties as a means to suppress noise in optical communications [1] and in interferometric [2] and mechanical bar gravity wave detectors [3–5]. The first experimental demonstration of squeezed light states followed shortly thereafter [6]. Many other groups have since demonstrated squeezed light using a variety of generation and detection techniques (see, e.g., Ref. [7] for a survey up to 1992). By contrast, there has been very little experimental work on squeezed states in *mechanical* systems; squeezed states have been demonstrated for a single, vibrating ion [8] and possibly for crystal phonons [9]; there have also been several theoretical proposals [10]. It would be of great interest to try to produce squeezed states for a mechanical oscillator structure much larger than a single atom, not only to test some of the ideas developed for the detection of very weak forces such as gravity waves (see, e.g., Ref. [11]), but also, at a more fundamental level, to extend the domain of manifestly quantum phenomena to macroscopic mechanical systems (see, e.g., Ref. [12] for a recent proposal to generate and detect quantum superpositions of spatially separated states in a macroscopic mechanical system).

One way to squeeze a mechanical oscillator initially in a thermal state would be to use parametric pumping, characterized by a term of the form $P(t)(a^{\dagger 2}+a^2)$ in the oscillator Hamiltonian (see, e.g., Ref. [5]). The first demonstration of this method for *classical* thermomechanical noise squeezing was performed by Rugar *et al.* [13] using a device comprising a cantilever several hundred microns in length and a few microns in crossection. The room temperature thermal vibrational motion in the fundamental flexural mode was parametrically squeezed in one quadrature to an effective temperature of about 100 K by periodically modulating the effective spring constant at twice the flexural frequency. A natural question to ask is whether *quantum* squeezing could be achieved in a similar device. In order to squeeze below the zero-point fluctuations, the thermal fluctuations of the cantilever before squeezing must not be too much larger in magnitude than the zero-point fluctuations [5]. Now, the lowest temperature to which a microdevice can be cooled using reasonably available equipment (e.g., a nuclear demagnetization cryostat) is around a mK. At such temperatures, we would require a cantilever with a fundamental frequency of around 100 MHz. A cantilever vibrating at radio frequencies might seem hopelessly unrealistic. However, recently Cleland *et al.* [14] succeeded in fabricating *micron*-sized, suspended Si beams with fundamental resonant frequencies of just this order. In this paper, we will show that substantial quantum squeezing can in fact be achieved using a cantilever device similar to that of Rugar *et al.* [13], but scaled-down to micron dimensions and with materials characteristics such as quality factor similar to those of the structures considered by Cleland *et al.* [14]. We will first discuss the method of generation and then follow with the method of detection.

The model structure which we consider is similar to the device of Rugar *et al.* [13], comprising a cantilever with one plate of a capacitor located on the cantilever surface and the other plate located on the substrate surface directly opposite. Unlike their device, however, the capacitor will serve as a component not only of the pump circuitry, but of the probe circuitry as well (more on this later). The classical equations of motion for the cantilever in the fundamental flexural mode are

$$\frac{d^2x}{dt^2}+\frac{\omega_c}{Q_c}\frac{dx}{dt}+\omega_c^2 x=\frac{q^2(t)}{2C_0 dm}+\frac{F_{\rm fluct}(t)}{m}, \qquad (1)$$

where, in terms of the pump voltage $V_p(t)$, the capacitor plate charge is

$$q(t)=\frac{C_0 V_p(t)}{1-x(t)/d}. \qquad (2)$$

The coordinate x denotes the displacement of the cantilever tip from the static equilibrium position ($V_p=0$), m is the cantilever effective mass, d is the equilibrium cantilever tip-substrate base separation, and C_0 is the capacitance for equilibrium separation d. Recall that, in terms of the frequency ω_c and relaxation time τ_c of the fundamental flexural mode, the quality factor is defined as $Q_c=\omega_c\tau_c$. Both Q_c and the random force term $F_{\rm fluct}$

model the effects of the thermal environment on the flexural mode. The c subscript, which denotes 'cantilever', is employed in order to distinguish the mechanical oscillator from the coupled probe electrical oscillator to be introduced later on.

Substituting (2) into (1) for pump voltage having the form $V_p(t) = V_0 \cos(\omega_p t + \phi)$ and assuming $|x| \ll d$, we obtain

$$m_c \frac{d^2x}{dt^2} + \frac{m_c \omega_c}{Q_c}\frac{dx}{dt} + [k_0 + k_p(t)]x \approx F_p(t) + F_{\text{fluct}}(t), \tag{3}$$

where $k_0 = m_c\omega_c^2 + \Delta k$, $\Delta k = C_0 V_0^2/2d^2$, $k_p(t) = \Delta k \cos(2\omega_p t + 2\phi)$, and $F_p(t) \equiv k_p(t)$. Note that the equilibrium static spring constant is shifted upwards by Δk. Thus, the resonant frequency of the cantilever is shifted to $\omega_c' = \sqrt{\omega_c^2 + \Delta k/m}$. There is also a shift downwards in the equilibrium position of the cantilever tip by the amount $C_0 V_0^2/(2dm_c\omega_c^2)$ and we have redefined the origin of x to coincide with this new equilibrium position. Note that one consequence of applying the pump voltage $V_p(t)$ across the capacitor is the sinusoidal modulation $k_p(t)$ of the spring constant. For phase $\phi = -\pi/4$, this modulation causes squeezing in the quadrature amplitude X_1 [5,13], where

$$\begin{aligned} X_1(t) &= x(t)\cos\omega_c' t - \omega_c'^{-1}\dot{x}(t)\sin\omega_c' t \\ X_2(t) &= x(t)\sin\omega_c' t + \omega_c'^{-1}\dot{x}(t)\cos\omega_c' t. \end{aligned} \tag{4}$$

Pumping the cantilever from an initial thermal equilibrium state at frequency $\omega_p = \omega_c'$, one obtains for the quantum uncertainty in X_1 [5]

$$\Delta X_1^2(t\to\infty) \approx \frac{\hbar}{2m\omega_c}(2n_T+1)\left(1 + \frac{Q_c\Delta k}{2m\omega_c^2}\right)^{-1}, \tag{5}$$

where $n_T = (e^{\hbar\omega_c/k_BT} - 1)^{-1}$. Note that we have replaced ω_c' with ω_c in (5) since this causes only a small error for the parameter values to be considered below. On the other hand, it is important to account for the frequency shift $\omega_c' - \omega_c$ when setting the pump frequency ω_p, since the resonance width ω_c'/Q_c can be smaller than this shift for large Q_c. In order to have quantum squeezing, we require that the squeezing factor $R = \Delta X_1/\sqrt{\hbar/2m\omega_c} < 1$, where recall that $\sqrt{\hbar/2m\omega_c}$ is the zero-point uncertainty. Thus, from (5) we have

$$R = \sqrt{\frac{2n_T+1}{1 + Q_c\Delta k/2m\omega_c^2}} < 1. \tag{6}$$

For illustrative purposes, we consider a crystalline sapphire cantilever with mass density $\rho = 3.99\times10^3$ kg/m^3 and assume the bulk value for Young's modulus: $E = 3.7\times10^{11}$ N/m^2. Sapphire is elastically isotropic to good approximation, thus simplifying the analysis. The preferred materials for experiment would probably be Si or GaAs. Substituting into (6) the expressions for the cantilever effective mass, $m = \rho lwt/4$, fundamental flexural frequency, $\omega_c = 3.516\sqrt{E/12\rho}\; t/l^2$, and capacitance, $C_0 = \epsilon_0\lambda lw/d$, we obtain the following conditions on the cantilever dimensions

$$\frac{t}{l^2} \lesssim \frac{1}{75} \tag{7}$$

$$\frac{l^4}{d^3t^3} \gtrsim 10^7, \tag{8}$$

where we have set $V_0 = 1$ V, $Q_c = 10^4$, $T = 1$ mK, and have expressed the various cantilever dimensions in units of micrometers. The parameter λ, which appears in the expression for the capacitance, is a geometrical factor to allow for the fact that the capacitor plate area need not coincide with the cantilever area. We have arbitrarily set $\lambda = 1/3$. Condition (7) follows from requiring that the thermal occupation number n_T be of order one or smaller, while condition (8) follows from requiring that the term appearing in the denominator in (6) be of order one or larger. We must have condition (7) as well since, if it did not hold, then n_T would be very large and unrealisable values for Q_c and the various cantilever dimensions would be required in order to compensate. Setting $l^2/t = 75$ and substituting into (8), we obtain $d^3t \lesssim 6\times10^{-4}$. As an example, the conditions are satisfied if we choose $d = 0.05$ μm, $t = 0.1$ μm, and $l = \sqrt{7.5} \approx 2.7$ μm. For these cantilever dimensions, the squeezing factor (6) is $R \approx 0.25$ and, thus, we have quantum squeezing. This factor is comparable with the best squeezing factors achieved for light [7] and corresponds to a noise reduction more than three orders of magnitude larger than that obtained in the recent phonon squeezing experiments of Garrett *et al.* [9].

If a cantilever could be realised having the same dimensions, but with quality factor $Q_c \approx 10^6$ instead of 10^4, then the squeezing factor would be $R \approx 0.025$, an order magnitude smaller. It clearly would be of great interest to determine whether micron-sized cantilevers could be fabricated with much higher quality factors. Little is currently known about the upper limits on Q_c. The few reported Q_c values for micron-sized cantilevers [14,15] are many orders of magnitude smaller than what can be achieved for large-scale mechanical resonators [16]. This is thought to be due to the increasing importance of surface defects for the dissipation of mechanical energy the larger the surface-to-volume ratio. Presumably, the surface defect density can be reduced with appropriate modifications of the fabrication process.

In the analysis above, it was assumed that the cantilever tip displacement x is much smaller in magnitude than the cantilever-substrate separation d. One might worry, however, that this is not the case for the very large electric fields resulting from applying a potential

difference of 1 V across a gap $d = 0.05\ \mu$m. Substituting the various parameter values into the expression for the equilibrium position shift, we obtain a displacement of about 3 Å which is much smaller than the cantilever length. We are therefore far from snapping the cantilever and well within the range of applicability of Hooke's force law. Similarly, the applied force $F_p(t)$ gives a displacement amplitude of about 1 Å for $\omega_p = \omega_c'$. Note that, if the frequency of the applied force was resonant with the frequency ω_c' (instead of being twice this frequency), then the displacement amplitude would increase by a factor $Q_c = 10^4$ to about 1 μm. We can now see that the displacement amplitude is small, despite the large applied electric field, because the applied force is off-resonance. It is also important to check that the applied force $F_p(t)$ is not resonant with a higher flexural mode of the cantilever. The frequency of the second flexural mode is about six times larger than the fundamental frequency and therefore the applied force is even further off-resonance with this mode. An estimate of the resulting displacement amplitude yields about 0.1 Å.

The Casimir force can also give rise to large deflections for submicron plate separations [17]. Using the expression for the Casimir force between two parallel plates of area A, $F_{\mathrm{casimir}} = \pi^2 \hbar c A/240 d^4$, we obtain a deflection on the order of an Angstrom for a cantilever with the above dimensions, including a width $w = 1\ \mu$m.

In the classical squeezing analysis of Rugar *et al.* [13], the analogous quantity to the squeezing factor (6) is the gain, defined as $G = |X|_{\text{pump on}}/|X|_{\text{pump off}}$, where $|X| = \sqrt{X_1^2 + X_2^2}$. The term $Q_c \Delta k/2m\omega_c^2$ [see (6)] also appears in their expression for G. However, there would appear to be a discrepancy: their solutions for $X_{\frac{1}{2}}(t)$ break down when this term exceeds one, hence restricting their squeezing maximum (minimum gain) to 1/2, whereas we have no upper bound on this term. The resolution lies in the fact that Rugar *et al.* assumed steady-state solutions. If this term exceeds one, as is the case for the parameter values we are considering, then $X_2(t)$ grows exponentially without bound as $t \to \infty$. Thus, the pumping should terminate after the characteristic time t_{ch} for which the squeezing factor largely reaches its limiting value (6), where [5]

$$t_{\mathrm{ch}}/\tau_c = \frac{2m\omega_c^2}{Q_c \Delta k} \ln\left(\frac{Q_c \Delta k}{2m\omega_c^2}\right). \tag{9}$$

A related issue concerns the conversion of mechanical energy into heat, possibly warming the cantilever sufficiently to take it out of the quantum squeezing regime. It is not clear whether the generated heat would dissipate sufficiently rapidly into the surrounding substrate to prevent this from happening; the heat dissipation rate clearly depends on the materials properties and layout of the device. Alternatively, heating will be negligible if the cantilever is pumped for a time smaller than the relaxation time. Therefore, we require $t_{\mathrm{ch}} < \tau_c$. Substituting into (9) the various chosen parameter values, we find $t_{\mathrm{ch}}/\tau_c \approx 0.1$. Thus, the limiting squeezing value can be largely attained without significant heating.

We now consider a possible way to measure the uncertainty ΔX_1. For the considered parameter values, the zero-point uncertainty is $\sqrt{\hbar/2m\omega_c} \approx 4 \times 10^{-14}$ m, while the squeezed uncertainty is $\approx 10^{-14}$ m. Measuring fluctuations of this small magnitude might at first appear a hopeless task. However, compare these numbers to the even smaller displacements of around 10^{-19} m which must be resolved for a metre long gravity wave bar detector. Remarkably, a displacement detector with sensitivity approaching 10^{-19} m was demonstrated as long ago as 1981 [18] (see also Sec. 10 of Ref. [16]). The detector was effectively an LC circuit, where changes in the capacitor plate separation are converted into changes in the resonant frequency $1/\sqrt{LC}$ of the circuit. This method is well-suited to our cantilever system, since the existing capacitor can also be used as a displacement sensor by forming part of an LC circuit. As an aside, note that capacitance-changes have also been used to detect cantilever deflection in an AFM (see, e.g., Ref. [19]).

Thus, we have in mind a two-stage process. In the pumping stage, the cantilever flexural mode vibrations are driven into a squeezed state as described above. After a time t_{ch}, pumping terminates and then the probe circuitry takes over. Forming an LC circuit with the existing capacitor, the cantilever equation of motion (1) becomes one of two coupled equations, where the other equation for the circuit charge q is

$$\frac{d^2 q}{dt^2} + \frac{\omega_e}{Q_e}\frac{dq}{dt} + \omega_e^2 q\left(1 - \frac{x}{d}\right) = \omega_e^2 C_0 \left[V_{\mathrm{pr}}(t) + V_{\mathrm{fluct}}(t)\right]. \tag{10}$$

The circuit resonant frequency is $\omega_e = 1/\sqrt{LC_0}$, where the e subscript stands for 'electrical'. We have also included a fluctuating voltage $V_{\mathrm{fluct}}(t)$, which describes the Johnson-Nyquist noise due to unavoidable circuit resistance $R = 1/(\omega_e Q_e C_0)$.

For a continuous measurement, it is essential that $X_1(t)$ is measured and not the ordinary displacement $x(t)$. In the latter case, the Heisenberg uncertainty principle would prevent one from measuring displacements below the zero-point fluctuations [11]. The quadrature $X_1(t)$ is measured for probe voltage satisfying $V_{\mathrm{pr}}(t) = V_0 \cos\omega_e t \cos\omega_c t$. This form couples q much more strongly to the quadrature amplitude X_1 than to X_2, provided $Q_e \omega_c/\omega_e \gg 1$ (see Sec. 10.7 of Ref. [11]). The necessary large quality factors Q_e can be achieved, for example, by using superconducting wires (see, e.g., Sec. 6 of Ref. [16]).

Simplifying (1) and (10) as in Sec. 10.5 of Ref. [11] to make them approximately linear in q and x, and then solving the corresponding quantum equations mo-

tion along the lines of Ref. [5], the uncertainty in the voltage across the capacitor for times $1/\omega_e \ll t \ll \min(\tau_c, \tau_e)$ is

$$\Delta V^2(t) \approx \frac{1}{8}\left(\frac{\Delta X_1(0)}{d}\right)^2 (V_0\omega_e t)^2 + \frac{\hbar\omega_e}{2C_0}\,(2n_T^e+1) + \frac{1}{24}\left(\frac{\hbar}{2m\omega_c d^2}\right)(V_0\omega_e t)^2\left(\frac{\omega_c t}{Q_c}\right)(2n_T^c+1)\,, \tag{11}$$

where $\Delta X_1(0)$ is the cantilever quadrature uncertainty at the start of the probe stage, defined as $t=0$. The circuit is assumed to be initially in thermal equilibrium, described by the distribution $n_T^e = (e^{\hbar\omega_e/k_BT}-1)^{-1}$. The right hand side of (11) involves three terms, the first of which depends directly on $\Delta X_1(0)$. This first term increases with time, eventually exceeding the second term which describes the voltage fluctuations across the capacitor due to the nonzero circuit resistance. On the other hand, the third term, which describes the return to thermal equilibrium of the cantilever because it is no longer being pumped, eventually exceeds the first term. Thus, the time interval over which the uncertainty $\Delta X_1(0)$ can be resolved is bounded both above and below by thermal noise. Choosing $\omega_e = \omega_c$, $Q_e > Q_c$, $V_0 = 1$ V, and the above considered cantilever parameter values, we find $10 < \omega_e t < 850$, with $\Delta V(t) \approx \Delta X_1(0)V_0\omega_e t/2\sqrt{2}d \approx 6\times10^{-5}$ V at, say, $\omega_e t = 800$ for $\Delta X_1(0) = 10^{-14}$ m. Again, if a cantilever with larger quality factor $Q_c \approx 10^6$ could be realised (and also $Q_e > Q_c$), then the upper bound would increase to 8.5×10^4, giving a much larger signal $\Delta V(t) \approx 6\times10^{-3}$ V at $\omega_e t = 8\times10^4$.

The uncertainty $\Delta X_1(0)$ is a quantum statistical quantity: the pump and probe stages must be repeated many times in order to accurately determine $\Delta X_1(0)$. Thus, we require not only frequency stability but also amplitude stability for the applied voltages $V_p(t)$ and $V_{\rm pr}(t)$. In particular, during the pump stage the shift in the equilibrium position of the cantilever tip due to uncontrollable fluctuations in $V_p(t)$ must be smaller than the squeezed magnitude 10^{-14} m obtained above. For the considered values, this imposes the requirement $|\delta V_p/V_p| < 10^{-5}$.

An implicit assumption of our analysis is that the capacitor plate on the substrate surface opposite the cantilever is rigidly fixed, with negligible fluctuations as compared with those of the cantilever tip—even in the squeezing regime. This must be checked, of course. We have obtained estimates of the surface fluctuations, modelling the capacitor plate/substrate surface as a half space [20]. Summing over all the different mode contributions (e.g., Rayleigh, mixed P-SV, etc.) to the surface fluctuations averaged over a plate area 1 μm^2, we find a fluctuation magnitude perpendicular to the surface of about 3×10^{-15} m at 1 mK. This figure is indeed smaller than the squeezed magnitude value 10^{-14} m. Note, however, that these surface fluctuations will prevent one from measuring squeezing amplitudes not much below this value for the current setup.

In conclusion, we have shown that substantial squeezing below the zero-point motion can be achieved for the flexural mode of a micron-sized cantilever. A method using a coupled LC circuit to measure the fluctuation amplitudes in the squeezing regime was also described. Displacement detectors with much better sensitivity than required were demonstrated some time ago [16,18], while cantilevers with the required dimensions and materials characteristics have recently been realised [14].

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[1] H. P. Yuen and J. H. Shapiro, IEEE Trans. Inf. Theory IT **24**, 657 (1978); J. H. Shapiro, H. P. Yuen, and J. A. Machado Mata, *ibid.* IT **25**, 179 (1979); H. P. Yuen and J. H. Shapiro, *ibid.* IT **26**, 78 (1980).

[2] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).

[3] J. N. Hollenhorst, Phys. Rev. D **19**, 1669 (1979).

[4] C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).

[5] L. P. Grishchuk and M. V. Sazhin, Zh. Eksp. Teor. Fiz. **84**, 1937 (1983) [Sov. Phys.—JETP **57**, 1128 (1983)].

[6] R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, Phys. Rev. Lett. **55**, 2409 (1985).

[7] H. J. Kimble, Phys. Rep. **219**, 227 (1992).

[8] D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **76**, 1796 (1996).

[9] G. A. Garrett, A. G. Rojo, A. K. Sood, J. F. Whitaker, and R. Merlin, Science **175**, 1638 (1997).

[10] X. Hu and F. Nori, Phys. Rev. B **53**, 2419 (1996); Phys. Rev. Lett. **76**, 2294 (1996); Phys. Rev. Lett. **79**, 4605 (1997); A. L. Burin, J. L. Birman, A. Bulatov, and H. Rabitz, cond-mat/9708113.

[11] V. B. Braginsky and F. Ya. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).

[12] S. Bose, K. Jacobs, and P. L. Knight, quant-ph/9712021.

[13] D. Rugar and P. Grütter, Phys. Rev. Lett. **67**, 699 (1991).

[14] A. N. Cleland and M. L. Roukes, Appl. Phys. Lett. **69**, 2653 (1996).

[15] T. D. Stowe, K. Yasumura, T. W. Kenny, D. Botkin, K. Wago, and D. Rugar, Appl. Phys. Lett. **71**, 288 (1997).

[16] V. B. Braginsky, V. P. Mitrofanov, and V. I. Panov, *Systems with Small Dissipation* (University of Chicago Press, Chicago, 1985).

[17] F. M. Serry, D. Walliser, and G. J. Maclay, J. Appl. Phys. **84**, 2501 (1998).

[18] V. B. Braginsky, V. I. Panov, and V. D. Popel'nyuk, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 423 (1981) [Sov. Phys.—JETP Lett. **33**, 405 (1981)].

[19] G. Neubauer, S. R. Cohen, G. M. McClelland, D. Horne, and C. M. Mate, Rev. Sci. Instrum. **61**, 2296 (1990).

[20] H. Ezawa, Ann. Phys. (N. Y.) **67**, 438 (1971).